Lidar inversion with variable backscatter/extinction ratios

James D. Klett

The conventional approach to solving the single-scattering lidar equation makes use of the assumption of a power law relation between backscatter and extinction with a fixed exponent and constant of proportionality. An alternative formulation is given herein which assumes the proportionality factor in the power law relationship is itself a function of range or extinction. The resulting lidar equation is solvable as before, and examples are given to show how even an approximate description of deviations from the power law form can yield an improved inversion solution for the extinction. A further generalization is given which includes the effects of a background of Rayleigh scatterers.

I. Introduction

The volume extinction $[\sigma(k)]$ and backscatter $[\beta(k)]$ coefficients at wave number k due to particulate scattering and absorption are given by integrals over the size distribution $n(r)$ of (assumed spherical) particles of radius r:

$$\sigma(k) = \pi \int r^2 Q(\tilde{m},kr)n(r)dr, \tag{1}$$

$$\beta(k) = \pi \int r^2 Q_\pi(\tilde{m},kr)n(r)dr, \tag{2}$$

where Q and Q_π are, respectively, the extinction and backscatter efficiencies, as determined by the Mie theory of scattering; both are functions of the complex index of refraction $\tilde{m}$ and size parameter kr. (Gaseous absorption contributions to σ are not included in this paper.) It is clear from their definition as weighted integrals over the size distribution that the ratio of backscatter to extinction will be, for a given wavelength and complex index, only a function of the shape of the distribution. (Shape here simply refers to the size distribution known to within an arbitrary scale factor.) Therefore, if the spectral shape and composition of the scattering aerosol are spatially invariant, so too will be the proportionality between β and σ. On the other hand, if the aerosol spectral form or composition does vary with location, β and σ can be regarded as having a proportionality which is range dependent.

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The traditional way to account for any dependence between β and σ is to assume a power law relationship of the form

$$\beta = Bo\sigma^k, \tag{3}$$

where Bo and k are constants, but k is no longer unity. Then the single-backscattering lidar equation may, with the use of Eq. (3), be reduced to the form

$$\frac{dS}{dr} = \frac{1}{\beta}\frac{d\beta}{dr} - 2\sigma, \tag{4}$$

$$= \frac{k}{\sigma}\frac{d\sigma}{dr} - 2\sigma, \tag{5}$$

with $S(r) = \ln[r^2P(r)]$, and where $P(r)$ is the power received from range r. The stable solution to Eq. (5) is

$$\sigma(r) = \frac{\exp[(S - Sm)/k]}{\left\{\sigma_m^{-1} + \frac{2}{k}\int_r^{r_m} \exp[(S - Sm)/k]dr'\right\}}, \tag{6}$$

where $Sm = S(r_m)$, $\sigma_m = \sigma(r_m)$, and $r \le r_m$.[1] In addition to requiring the validity of Eq. (3) and the absence of multiple scattering, Eq. (6) also assumes that gaseous absorption effects are negligible.

From the discussion preceding Eq. (3) it is clear that it would be more realistic to regard β and σ as linearly related but with a proportionality which is itself a function of range or extinction. An approximate and simple way of formulating the inversion problem in this fashion is given in this paper. Any available information concerning deviations from a strict linear relationship can be easily incorporated into the solution implementation. An example of the backscatter/extinction variation for water clouds is given, and it is shown how even a very approximate description of this variation can yield an improved inversion solution for the extinction. Finally, a further generalization is carried out which includes the effects of a background of Rayleigh scatterers.

II. Modified Inversion Solution for Variable B

Suppose now that β and σ are related according to

$$\beta = B\sigma^k, \tag{7}$$

where B is a function of r, and k is a constant. (From the discussion above one would generally set $k = 1$, but it is retained here as a fixed parameter primarily to facilitate direct comparison below between new and old expressions.) Then on substitution of this expression into Eq. (4) one finds

$$\frac{dS}{dr} = \frac{1}{B}\frac{dB}{dr} + \frac{k}{\sigma}\frac{d\sigma}{dr} - 2\sigma, \tag{8}$$

or

$$\frac{d(S - \ln B)}{dr} = \frac{k}{\sigma}\frac{d\sigma}{dr} - 2\sigma. \tag{9}$$

But this is the same as Eq. (5) except that S is now replaced by $S - \ln B$. Hence one can write down the solution immediately in the form of Eq. (6):

$$\sigma(r) = \frac{\exp[(S - Sm - \ln B + \ln Bm)/k]}{\left\{\sigma_m^{-1} + \frac{2}{k}\int_r^{r_m} \exp[(S - Sm - \ln B + \ln Bm)/k]dr'\right\}} \tag{10}$$

or

$$\sigma(r) = \frac{(Bm/B)^{1/k}\exp[(S - Sm)/k]}{\left\{\sigma_m^{-1} + \frac{2}{k}\int_r^{r_m} (Bm/B)^{1/k}\exp[(S - Sm)/k]dr'\right\}}, \tag{11}$$

where $Bm = B(r_m)$.

An apparent difficulty with Eq. (11) is that in general we cannot expect to know $B(r)$ precisely until we also know $\sigma(r)$; i.e., it is really part of the solution. However, an approximate solution to Eq. (11), still more accurate than the default version with $Bm/B = 1$ [i.e., Eq. (6)], can be obtained if there is available at least some information on the dependence of B on either r or σ. Suppose, for example, that for the aerosol in question there has been established an approximate empirical correlation of the form $B = f(\sigma)$. Then an approximate solution can be obtained by a two (or more if needed)-stage iteration: First, let $Bm/B = 1$ and solve Eq. (11) or (6) to obtain the first-order solution σ_1. Second, solve Eq. (11) for the second-order solution σ_2, where now we use

$$B(r) = f[\sigma_1(r)]. \tag{12}$$

We shall now turn to an example of this approach and compare the outcome with the familiar method based on Eqs. (3) and (6).

III. Inversion Examples for Conditions of Fog and Low Clouds

An example of the variability of B with range for conditions of fog and low clouds is shown in Fig. 1. The data displayed (supplied by E. Measures, Atmospheric Sciences Laboratory, White Sands Missile Range, N.M. 88002, personal communication) derives from *in situ* measurements made by balloon-borne particle counters in Meppen, Germany, in the fall of 1980.[2,3] Mie theory was applied at a wavelength of 1.06 μm to obtain profiles with height of β and σ under the assumption that the

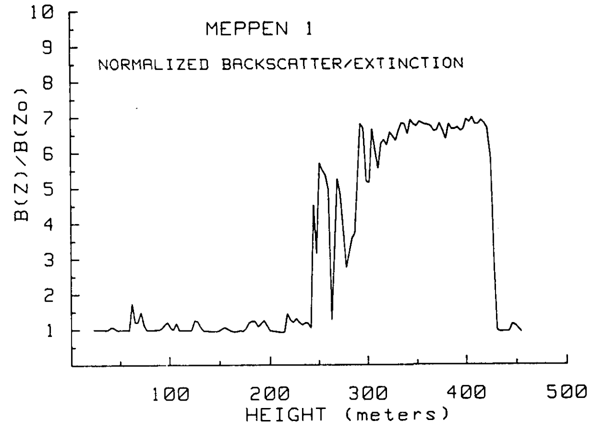

Fig. 1. Dependence on range of backscatter-to-extinction ratio for low stratus cloud in Meppen, Germany, 1980. Ratio normalized by value at near range point Zo. Data based on *in situ* measurements of drop size distributions. Light wavelength = 1.06 μm (adapted from calculations of E. Measures, ASL/WSMR; personal communication).

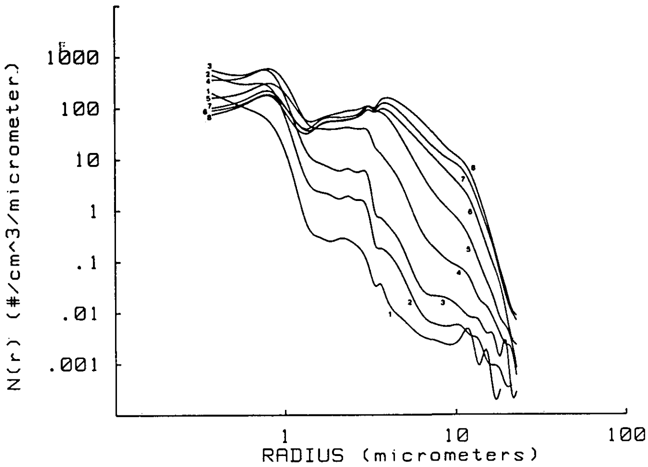

Fig. 2. Drop size distributions from in-cloud measurements in Meppen, Germany, 1980. The eight distributions shown are averages for the following labeled ranges of extinction (km): 1, (0.1,0.6); 2, (0.6,1.5); 3, (1.5,3); 4, (3,15); 5, (15,40); 6, (40,65); 7, (65,100); 8, (100,200) (from Ref. 3).

particles were spherical drops of pure water. A set of characteristic aerosol spectra obtained by these measurements for various extinction intervals is shown in Fig. 2.[3]

It is evident in Fig. 2 that the data may be somewhat incomplete, as the distributions are truncated at small sizes where the particle concentrations are still high. This spectral truncation is due to limited instrumental resolution. In addition, one should perhaps take into account the variation of $\tilde{m}$ with size; for example, the presence of soot particles in the smaller droplets may significantly affect their optical properties.[4,5] But in spite of these limitations, the data shown serve well to indicate the kind of behavior that is expected to occur typically. The trends shown in Fig. 2 are, for example, quite consistent with what one expects for water clouds,

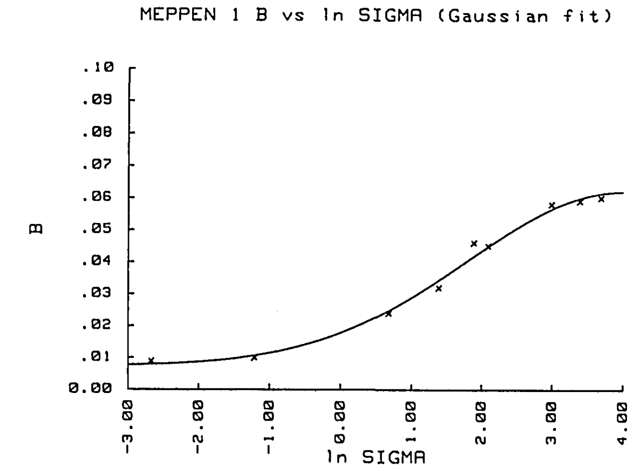

Fig. 3. Log-Gaussian fit [Eq. (13)] of backscatter to extinction ratio vs extinction for Meppen data.

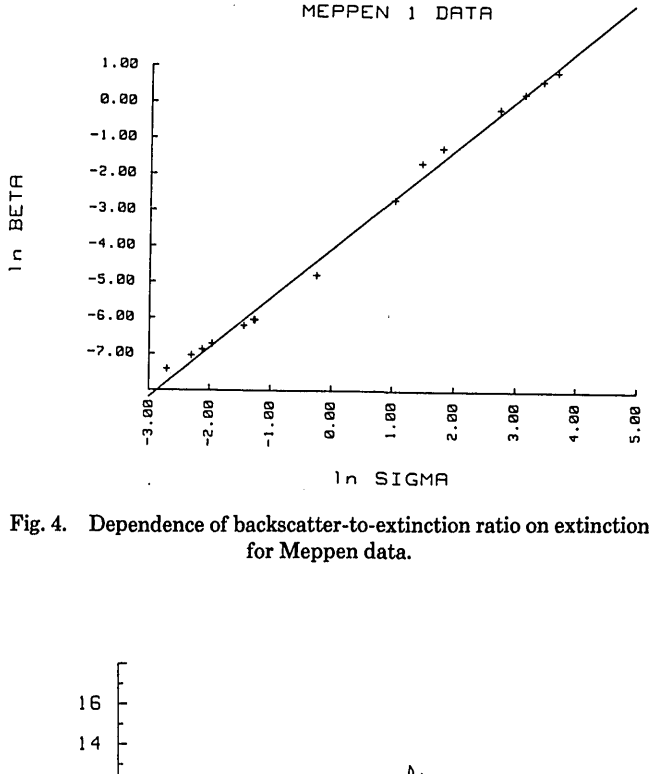

Fig. 4. Dependence of backscatter-to-extinction ratio on extinction for Meppen data.

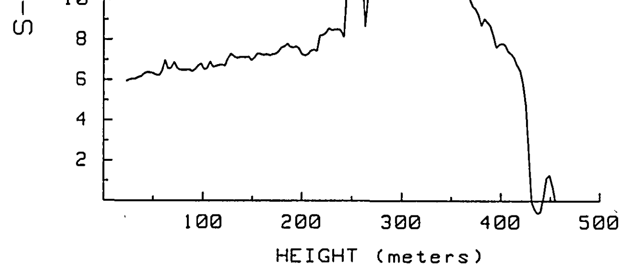

Fig. 5. Simulated relative range-corrected logarithmic signal, $S - Sm$ [Eq. (14)], for Meppen data.

namely, that the drop spectrum will evolve toward a bimodal shape and a larger average drop size with increasing liquid water content.[6] The continuing flow of water mass up the size spectrum to larger drops is also expected to be positively correlated with increasing extinction, especially if condensation continues to replenish the spectrum at small drop sizes. Such continued production of small drops by condensation and diffusional growth is to be expected for actively growing clouds with an evolving bimodal drop distribution as long as an upward flux of moist air continues.

From the profiles of β and σ described above one can also obtain an empirical correlation between B and σ. This is shown in Fig. 3, where a lognormal curve has been fitted to a total of nine data pairs; the equation of the curve is

$$f(\sigma) = 0.0074 + 0.055 \exp\{-[(\ln\sigma - 4)/3.1]^2\}. \tag{13}$$

One may alternatively use linear regression applied to a plot of β and σ to find values for Bo and k in the power law expression of Eq. (3). The best fit line is shown in Fig. 4 and has the parameters $Bo = 0.017$ and $k = 1.34$. The basis for the departure from $k = 1$ can be discerned by comparing Figs. 2 and 4. The plotted data in the latter figure tend to trace out a slightly undulating curve, the ends of which have a smaller slope (k near unity) than the middle section. This corresponds qualitatively to the similar unimodal shapes of spectra 1–3 and the similar bimodal shapes of spectra 6–8 in Fig. 2. Such behavior is consistent with the expectation that $k = 1$ over any interval where the spectral shape is invariant, as discussed above. In general, it is the changing proportionality between β and σ with changing spectral composition or shape that causes any overall power law description of $\beta(\sigma)$ to produce $k \neq 1$.

By integration of Eq. (4) the relative signal, $S - Sm$, appearing in the inversion solution may be obtained as a function of β and σ:

$$S - Sm = \ln\frac{\beta}{\beta_m} + 2\int_r^{r_m} \sigma dr', \tag{14}$$

where $\beta_m = \beta(r_m)$. Then by substituting the calculated profiles of $\beta(r)$ and $\sigma(r)$ for the Meppen data, a simulation of the relative signal is available (Fig. 5) from which one may in turn use Eq. (6) or (11) to attempt to retrieve $\sigma(r)$ for various choices of k or B. For the purpose of this paper the correct value for σ_m will be assumed as a given quantity. (The sometimes formidable but separate problem of estimating σ_m lies outside the scope of this paper.)

Let us first consider use of Eq. (6) with $k = 1$ or 1.3. The inversion results shown in Figs. 6 and 7 demonstrate that the better choice is $k = 1.3$, as expected from the previous fit of the data to the power law form. (Results like those shown in Figs. 6 and 7 have been obtained previously and independently from the Meppen data by E. Measures, ASL/WSMR; private communication.) The solution for $k = 1$ is seen to overestimate the extinction by nearly an order of magnitude in the first 250 m, where the visibility is high.

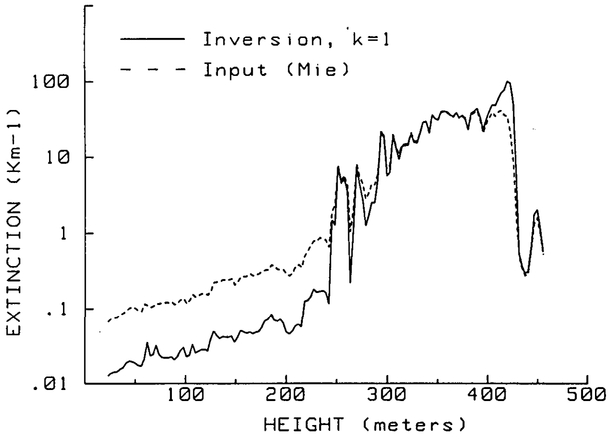

Fig. 6. For Meppen data, comparison of input profile of extinction, based on Mie theory computations over measured size distributions to inversion solution [Eq. (6)] with $k = 1$.

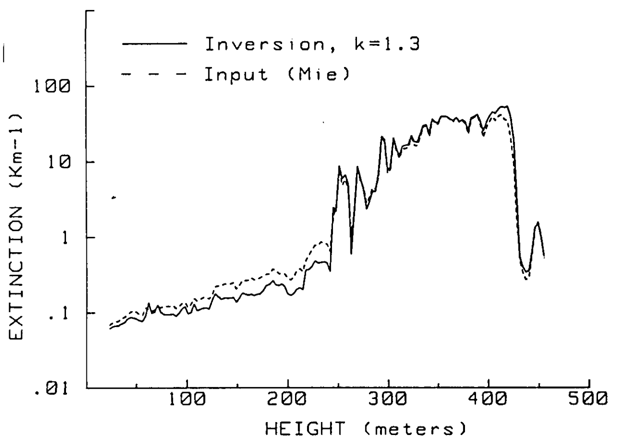

Fig. 7. Same as Fig. 6 but with $k = 1.3$.

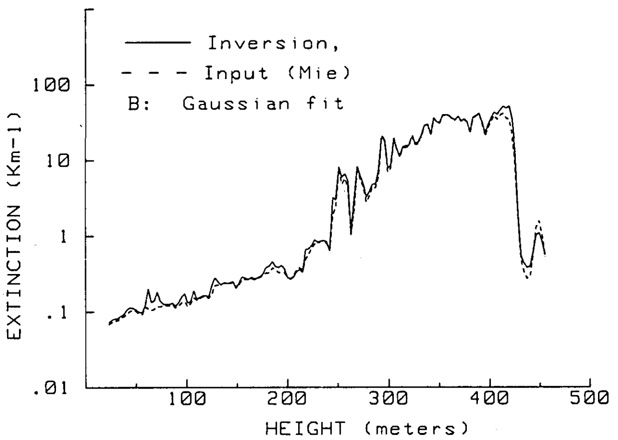

Fig. 8. For Meppen data, comparison of input and inversion profiles of extinction. Inversion via Eq. (11) with B from Eq. (13) and $k = 1$.

The inversion using Eq. (11) with $k = 1$ and B given by Eq. (13) is shown in Fig. 8. The agreement with the actual extinction profile is seen to be considerably better than was possible with either of the inversions based on Eq. (6). The corresponding estimate of $B(r)$ from Eq. (13) also agrees well with the actual profile, as shown in Fig. 9. The discrepancies that do occur can be attributed in part to the limited information content of the nine data pairs on which Eq. (13) is based and also in part to the fact that the relationship between B and σ is not quite unique in principle: The procedure of representing $B(r)$ by $f[\sigma(r)]$ would be mathematically proper if there were a one-to-one transformation between B and σ. But in general it is in fact many-one, as more than one value of B may occur, at different locations for the same value of σ. However, in practice such sources of error would likely prove less important than a host of other uncertainties which arise due to factors such as noise, multiple scattering, gaseous absorption, instrumental problems, and especially errors in σ_m.

One might also attempt to improve the inversion solutions by letting the exponent k become a function of range or extinction instead of a fixed constant. For example, one might choose $k = 1$ over intervals where the drop spectral form is thought to be relatively fixed and $k = 1.3$ over the transition region from one characteristic spectral shape to another (e.g., unimodal to bimodal). However, once k ceases to be a constant, the inversion solutions Eq. (6) or (11) no longer are valid, as the passage from Eq. (4) to (5) or (8) can no longer be made unless terms describing gradients in k are also included. Numerical experiments show that these extra terms are significant, as the ploy of merely letting k become a variable in Eq. (6) or (11) produces very poor results.

IV. Inclusion of Effects of Rayleigh Scattering

An extension of the inversion solution Eq. (6) to include the effects of a background of known molecular or Rayleigh scatterers has been carried out recently by

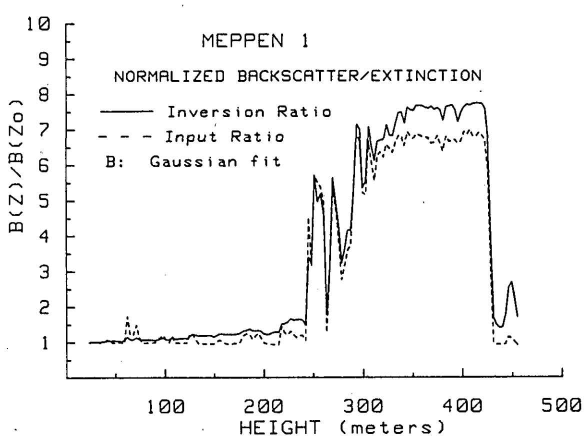

Fig. 9. For Meppen data, comparison of input and inversion profiles of backscatter-to-extinction ratio. Inversion via Eq. (11) with B from Eq. (13) and $k = 1$.

Fernald.[7] In this section a similar extension of the modified inversion solution Eq. (11) will be developed.

Let the total backscatter coefficient be expressed as

$$\beta(r) = B_P(r)\sigma_P(r) + B_R\sigma_R(z), \tag{15}$$

with

$$B_R = 3/(8\pi), \tag{16}$$

$$B_P(r) \approx f[\sigma_P(r)]. \tag{17}$$

Subscripts P and R refer to the aerosol particle and Rayleigh contributions, respectively. Thus Eq. (16) describes the fixed backscatter-to-extinction ratio for Rayleigh scatterers, while Eq. (17) describes the ratio for the particles according to formulation of Sec. II, with the function $f(\sigma_P)$ being the empirical connection between B_P and σ_P. Also the Rayleigh extinction coefficient $\sigma_R(z)$ is assumed to be a known function of height z.

On substituting Eq. (15) into the basic lidar differential equation, Eq. (4), and noting also that now $\sigma = \sigma_P + \sigma_R$, we obtain

$$\frac{dS}{dr} = \frac{1}{\beta}\frac{d\beta}{dr} - 2(B_P^{-1}\beta_P + B_R^{-1}\beta_R), \tag{18}$$

$$= \frac{1}{\beta}\frac{d\beta}{dr} - 2B_P^{-1}\beta + 2(B_P^{-1} - B_R^{-1})\beta_R. \tag{19}$$

Therefore, if one defines a new signal variable S' as

$$S' - Sm' = S - Sm + \frac{2}{B_R}\int_r^{r_m} \beta_R dr' - 2\int_r^{r_m} \frac{\beta_R dr'}{B_P}, \tag{20}$$

Eq. (19) can be rearranged to read as follows:

$$\frac{dS'}{dr} = \frac{1}{\beta}\frac{d\beta}{dr} - \frac{2\beta}{B_P}. \tag{21}$$

But this has the same form as Eq. (5), except for the presence of the function $B_P(r)$. The solution to Eq. (21) can, therefore, be constructed in the same fashion as Eq. (6) to obtain

$$\beta(r) = \frac{\exp(S' - Sm')}{\left[\beta_m^{-1} + 2\int_r^{r_m} \frac{\exp(S' - Sm')dr'}{B_P}\right]}, \tag{22}$$

Eqs. (20) and (22) comprise the solution to the problem stated at the beginning of the section. Note that if $\beta_R \to 0$, Eq. (11) is recovered (with $k = 1$). Similarly, if B_P is a constant, Eq. (22) becomes equivalent to the solution obtained by Fernald.[7]

Implementation of Eqs. (20) and (22) would parallel the procedures already described. Examples of the application of this model will be presented elsewhere; here the goal has been merely to present formally this particular generalization to show how a known background level of scattering could be included in the formulation of Sec. II.

V. Discussion

A new theoretical framework has been presented for dealing with the problem of variations in the backscatter-to-extinction ratio that occur because of variations in aerosol particle composition or spectral shape. It has been shown that if this ratio can be empirically correlated with extinction magnitude, this information can then be incorporated into a modified stable inversion solution to improve the accuracy of retrieved extinction profiles. With this physically direct and simple approach the backscatter/extinction ratio becomes a modulating factor in an otherwise linear relationship between backscatter and extinction.

The familiar alternative approach, whereby information on backscatter and extinction is cast in the form of an empirical power law relationship, has a less direct physical justification but nevertheless also has practical value with respect to the accuracy of retrieved profiles of extinction so long as the exponent is a constant representative of all the empirical data. However, attempts to refine the inversion solutions by letting the exponent become a variable parameter are ill-advised, as the available solution forms no longer apply, and serious errors may result.

An example has been given of an aerosol for which the backscatter/extinction ratio could be described approximately as a function of extinction [Eq. (13) and Fig. 3]. This was provided as an illustration of the technique of incorporating information on departures from a linear relationship between β and σ to invoke the new inversion solution. The functional form of the particular example chosen was, of course, not intended to be viewed as a universally applicable relationship. Nevertheless, some of its shape characteristics were found to be qualitatively as expected, and similar reasoning based on such factors as meteorological conditions, or the source and age of the aerosol, might similarly provide some insight into the approximate deviations from a linear relationship between β and σ in various real applications.

It should also be noted that great precision in the functional description for β/σ is not required to obtain useful results. As many numerical simulations have shown, even if nothing is known about the aerosol, the standard default assumption that β and σ are proportional may be used to obtain extinction profiles that often reflect surprisingly well the qualitative and quantitative trends of the actual distributions. This is fundamentally a consequence of the stability of the solution form given by Eq. (6). (The contrary assertion that β/σ must be very accurately known to get any useful information on σ is still made occasionally, but this is an erroneous notion that first came about from studies based on the unstable form of the lidar inversion solution. The hazards of that approach in general and in the context of the relationship between backscatter and extinction have been discussed elsewhere.[1])

Mathematically speaking, the present formulation relaxes the previous requirement that any account, theoretical or experimental, of the relationship between backscatter and extinction must be cast into an approximate power law form to make use of the standard inversion solution. With Eq. (11) any functional form, $\beta = f'(\sigma)$ can be used to obtain quickly an approximate solution for σ by substituting $B = f'(\sigma)/\sigma$ in Eq. (11) and iterating a few times as described in Sec. II. As a simple example, the usual solution based on the power law form of Eq. (3) may be recovered by substituting $B = B_0\sigma^{k-1}$ into Eq. (11) and iterating.

Finally, the numerical examples given in this paper suggest that for the same amount of aerosol scattering information, the modulated backscatter/extinction ratio approach gives better results than the power law formulation. Further study of the new modified inversion solution will of course be required to assess more fully its practical utility.

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References

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Meetings Calendar *continued from page 1584*

1985
June

24–28 Optical Signal Processing course, Troy *Off. of Continuing Studies, Rensselaer Polytechnic Inst., Troy, N.Y. 12180*

24–29 Fourier & Computerized Infrared Spectroscopy Int. Conf., Ottawa *Natl. Res. Council of Canada, L. Baignee, Conf. Services Off., Ottawa, Ontario, Canada K1A OR6*

24–5 July Applied Optics Summer course, London *J. Dainty, Optics Sec., Blackett Lab., Imperial Coll., London SW7 2BZ, England*

24–5 July Applied Materials Technology: Materials Processing for Process-Sensitive Manufacturing course, Edinburgh *Off. of Summer Session, Rm. E19-356, MIT, Cambridge, Mass. 02139*

25–27 11th Int. Symp. on Machine Processing of Remotely Sensed Data, West Lafayette *D. Morrison, Purdue U./LARS, 1291 Cumberland Ave., West Lafayette, Ind. 47906*

26–28 Int. Congr. on Lasers in Medicine & Surgery, Bologna *Medicina Viva, Viale dei Mille, 140, 43100 Parma, Italy*

26–28 Local Area Networks course, Wash., D.C. *Data-Tech Inst., Lakeview Plaza, P.O. Box 2429, Clifton, N.J. 07015*

July

1–4 Int. Conf. on Dynamical Processes in Excited States of Solids, Villeurbanne *W. Yen, U. of Wisconsin, Physics Dept., 1150 University Ave., Madison, Wisc. 53706*

3–4 Introduction to Military Thermal Imaging course, Kent *SIRA, Ltd. Conf. Unit, South Hill, Chislehurst, Ken BR7 5EH, England*

8–10 Flow Visualization Techniques: Principles & Applications course, Ann Arbor *Eng. Summer Confs., 200 Chrysler Ctr., N. Campus, U. of Mich., Ann Arbor, Mich. 48109*

8–11 3rd Conf. on Coherent Laser Radar: Technology & Applications, Malvern *J. Vaughan, Royal Signals & Radar Establishment, Ministry of Defense, St. Andrews Rd., PD316, Great Malvern, Worcestershire, WR14 3PS, U.K.*

8–12 17th Int. Conf. on Phenomena in Ionized Glasses, Budapest *I. Abonyi Roland Eotvos Physical Soc., P.O. Box 240, 1368 Budapest, Hungary*

8–12 Synthetic Aperture Radar Technology & Applications course, Ann Arbor *Eng. Summer Confs., 200 Chrysler Ctr., N. Campus, U. of Mich., Ann Arbor, Mich. 48109*

8–12 Laser Fundamentals & Systems course, Wash., D.C. *Eng. Tech., Inc., P.O. Box 8859, Waco, Tex. 76714*

8–12 Infrared Spectroscopy: Interpretation of Spectra course, Brunswick *D. Mayo, Chem. Dept., Bowdoin Coll., Brunswick, Me. 04011*

8–12 Quality Control for Photographic Processing course, Rochester *J. Compton, RIT, P.O. Box 9887, Rochester, N.Y. 14623*

8–12 Laser Fundamentals & Systems course, Dallas *Eng. Tech., Inc., P.O. Box 8859, Waco, Tex. 76714*

8–19 Lasers & Optics for Applications course, Cambridge *MIT, Res. Lab. of Electronics, Cambridge, Mass. 02139*

9 ISDN Markets course, Arlington *Communications & Info. Inst., c/o Info. Gatekeepers, Inc., 214 Harvard Ave., Boston, Mass. 02134*

continued on page 1656